# Why are small firms more likely to use convertible debt?

Ming-Jin Jiang[*†]
Universidad Carlos III de Madrid

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**Abstract**

Non-verifiability of firm returns may lead to firms' ex-post strategical default. Under non-verifiability of firm returns, a principal-agent model is presented in this paper. Firms with heterogeneous initial assets need funds to invest in a project whose return depends on the fund invested and firms' effort, which is unobservable to the financiers. We show that convertible debt contracts can mitigate ex-post inefficiency. Moreover, we further generalize the model into two directions: moral hazard and risk aversion of the firms. In both cases, we prove the optimality of convertible debt contracts for small firms.

Keywords: Optimal contracts, Non-verifiable firm returns, Moral hazard, Risk aversion

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## 1 Introduction

Consider a firm that needs to raise funds in order to invest in a project. In the literature of firms' choices of financing sources, it is well known that the conflicts

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[†]Contact information: mjiang@eco.uc3m.es

between the firms[1] and their financiers may cause economic inefficiency. In the firms with debt financing, firms might choose to invest in too risky projects, or they might hide the cash flows and default on their debt even they are able to pay back. If the firms are risk neutral, debt financing can effectively prevent the owner-managers to shirk (Innes (1990)). On the other hand, if firms choose equity financing, they might exert too little effort (Dybvig and Wang (2002)). However, the firms do not have incentive to hide their cash flows. Therefore, whether a firm should choose debt or equity financing depends on which of the two incentive problems is relatively more severe.

We start with an environment in which firms exert observable effort and they are protected by the limited liability when they default. Firms' ex-post returns are observable to both firms' owner-managers and their financial claimants, but are not verifiable by a third party (i.e., court). It is possible that non-verifiable firm returns, which results in firms' strategic defaults, distort the investors' efficient lending decisions. We consider firms are heterogeneous in their initial assets. The firms' benefits of defaulting strategically varies with their initial assets. Firms with smaller initial assets are more likely to default strategically than those with larger initial assets. The reason is that small firms have little to lose if they default and file for bankruptcy. If this problem is serious, compared with debt financing, equity financing may be preferable. Afterwards, we relax the assumption of observable effort and consider unobservable effort. Under this setting, another agency problem arises with equity financing, the incentives of firms' owner-managers to exert efficient effort are distorted under equity financing.

Our goal is to derive the optimal contracts such that in equilibrium, firms exert efficient effort and there are no strategic defaults. In this paper, we show that convertible debt contracts are optimal. Such a contract gives the holders an unilateral right to convert the debt into equity at the predetermined time and price (conversion rate). Convertible debt has the properties that combine both debt and equity. Specifically, it can thought as a standard debt contract plus a call option to convert the debt into equity when the firm's return has greater upside potential. We use this feature of convertible debt and our result shows that the problem of firms' strategic defaults due to non-verifiable firm returns can be solved.

The available evidence about convertible debt shows that small firms (i.e., firms with smaller initial assets) and firms with higher profit-to-assets ratio are more likely to use convertible debt over standard debt than larger firms. Noddings, Christoph and Noddings (2001) analyze the trading of U.S. convertible debts and convertible

[1] For simplicity, we assume firms' owners are also managers of the firms. Hence, we can ignore the conflict between the owners and the managers of the firms.

preferred stocks in January 2000. They find that among the total of 311 companies that use convertible debts, 58% are micro or small firms. Kahan and Yermack (1998) show empirically that the issuance of convertible debts is negative significantly related to the firm size[2]. Lewis, Rogalski and Seward (1999) find that firms with higher profit-to-assets ratio are more likely to use convertible debt.

Our paper is related to the literature on optimal contracts when the credit market is imperfect. In particular, Innes (1990) shows that, under limited liability and moral hazard due to the firms' unobservable effort, standard debt contracts are optimal. Since firms are residual claimers under standard debt contracts, standard debt contracts give firms incentives to exert effort. However, Innes (1990) considers the environment with verifiable firms' cash flows. We instead, relax the assumption of verifiable firm returns and characterize the optimal contracts assuming that the firm returns are not verifiable.

Besides, this paper is also closely related to the literature of incomplete contracts pioneered by Hart (Hart (2001), Hart and Moore (2007)). Due to the non-verifiability of firm returns, contracts can not be contingent on the firms' returns. As Hart mentioned, incompleteness of contracts open the door to a theory of ownership. In a recent work by Fluck (2010), he mentions that when the firms' returns or the owner-managers' misbehavior can not be verified or are too costly to be verified by the court, there are at least two ways to accomplish the optimal contracts. One is to make the contract contingent on other verifiable terms. The other way is to grant the investors an unconditional control right. This right allows the investors to threat the firm's owner-manager to replace him/her or to liquidate the firm's assets even though the firm's return is in the upside. However, this threat only works if the project is long-term. In the last period of the project, the owner-manager can never be induced to make the repayment without defaulting. Moreover, this threat only is effective if the firm has substantial assets. For the firms with little assets, this threat is not effective since small firms have little to lose. Consequently, the financiers are not willing to lend to small firms even though small firms have projects with positive present values. This leads to an *ex-ante* inefficiency. Hence, in order to discourage firms' strategic default *ex-post*, *ex-ante* inefficiency must be sacrificed at least partially.

Another related strand of literature focuses on the optimal security design of venture capital financing. It studies the agency problem caused by double moral hazard between the firms and the venture capitalists. Repullo and Suarez (RS,

[2]In their paper, firm sizes are defined by measuring firms' total initial assets. In particular, micro-cap: smaller than 225 million, small-cap: between 225 million and 1.25 billion, medium-cap: bewteen 1.25 billion and 10.5 billion, large-cap: larger than 10.5 billion.

2004) and Schmidt (2003) show that under a double moral hazard problem, the optimal contracts are convertible debt contracts. In RS's papers, first, convertible debt contracts solve the agency problem because convertible debt contracts allow the venture capitalists to share the firms' profit and hence provide the venture capitalists incentives to exert effort. Second, by using stage financing (which is commonly used in venture capital financing), the venture capitalists can threat the firm to stop providing them credits in order to induce the firms to exert effort. Schmidt (2003) shows that convertible debt contracts can induce both venture capital firms and venture capitalists to exert effort sequentially. In his model, both firms of all different size and the venture capitalists will only exert effort under convertible debt contracts.

In this paper, the impact of non-verifiable firm returns are crucial on determining the optimal contracts. We first construct a simple model to show that standard debt contracts are dominated by both equity and convertible contracts for small firms because small firms have incentives to default strategically if firms' returns are not verifiable. Furthermore, we generalize the model in two directions. First, we consider a moral hazard problem due to firms' unobservable effort. Second, instead of assuming risk-neutrality of both firms and their financiers, we consider risk aversion the firms.

In the former case (with moral hazard probelm), we derive the optimality of convertible debt contracts for small firms. This result is consistent with the empirical evidence (Noddings, Christoph and Noddings (2001) & Lewis, Rogalski and Seward (1999)). Moreover, under this setting, we show that the probability of using convertible debt is positively related with the firms' profit-to-asset ratio, which is also found in the data (Lewis, Rogalski and Seward (1999)).

In the latter (with risk-averse firms), our results suggest that, under certain conditions, in partucular, if the firms' utility function has a complete monotone first derivative and if the probability of failure of the project under equity contracts is lower than $\frac{1}{2}$, convertible debt contracts are optimal for small firms if firms are risk-averse. The reason is that small firms have incentives to default strategically under standard debt contracts. Hence, standard debt contracts are dominated. Moreover, under the condition of the probability of failure of the projects being low, convertible debt contracts dominates equity contracts because convertible debt achieves better risk sharing. As for large firms, since they do not have incentives to default strategically, both standard debt and convertible debt contracts are optimal.

We also discuss the case when both directions of generalizations exist at the same time. We conclude that the relation between moral hazard and the relative degree of risk aversion of the firms and the financiers importantly shape the optimal contracts. In particular, we conjecture that the result of the optimality of convertible debt contracts for small firms still holds even when considering the moral hazard problem and the risk aversion of the firms.

The outline of this paper is the following: In Section 2, we first analyze the benchmark model. Afterwards, we relax the assumption of verifiability of firm returns and characterize the optimal contracts. In Section 3 and Section 4, we generalize the model in two directions – moral hazard and risk aversion – and analyze the optimal contracts under each case. We further discuss the more general case when considering both directions together in Section 5. Finally, we conclude in Section 6.

# 2 The Model

In this paper, a principal-agent model is presented. Firms (agents) have an investment project of positive present value, but they do not have funds to finance the project. As a result, firms have to obtain the funds from the financiers, the principals. The assumptions of the model are:

**Assumption 1** Firms and financiers are both risk neutral. The firms are heterogenous in terms of their initial assets[3] $A$, $A \in \left(0, \overline{A}\right)$, where $\overline{A}$ is sufficiently high. The financiers' opportunity cost of lending per unit is assumed to be exogenous, and for simplicity, $i = 1$.

**Assumption 2** The investment project yields a random return

$$y = \begin{cases} \theta & \text{if the project succeeds} \\ 0 & \text{if the project fails} \end{cases}$$

The distribution of the realized returns is endogenous, depending on the funds $B$ invested in the project and firms' effort $e$ which is observable to the financiers. For simplicity, we assume that there are two levels of effort, $e \in \{e_H, e_L\}$. The cost is increasing in the effort with $c(e_L) = 0$ and $c(e_H) = c_H > 0$.

The probability of success of the project is denoted as $p(B, e)$, with $p'_B(B, e) > 0$ and $p''_B(B, e) < 0$ for any $e$, and $p(B, e_H) > p(B, e_L)$ for any $B$.

**Assumption 3** The project returns of firms are observable to both parties (firms and their financiers), however, returns are not verifiable by a third party (e.g., court).

**Assumption 4** Firms' liability to debt, as well as the financiers' liability to the investment are limited. That is, if a firm obtains debt from the financier, once

[3]We assume the firms' initial assets can be liquidated without any liquidation cost. Firms can liquidate (partially) their own assets and self finance. Under the assumption of no liquidation cost, the firms' are indifferent between self-financing or external financing.

it defaults and files for bankruptcy, the financier (lender) liquidates the firm's assets up to the required repayment. Besides, the repayment to the financiers whether the project succeeds or not is non-negative.

**Assumption 5** Financiers compete *a la* Bertrand.

Assumption 3 is crucial in this paper. In the following analysis, we first analyze the benchmark model in which Assumption 3 is ignored. In other word, in the benchmark model, we consider an environment in which firm returns are verifiable. Next, we analyze the effect of non-verifiability on the optimal contracts. Further, we generalize the model by relaxing Assumption 2 and Assumption 1, respectively. In particular, in the first generalized model, we consider that firms' effort is unobservable to the *financiers.* In the second generalized model, we consider risk averse firms and risk neutral *financiers*.

## 2.1 The benchmark case

In this section, we derive the optimal contract as a benchmark in the environment that both, firms' effort and project returns, are observable and verifiable. Afterwards, we focus on three types of contracts which are commonly used in reality: (1) standard debt contract, (2) equity contract and (3) convertible debt contract.

The optimal contract outcomes are the solution to the following problem:

$$\max_{B,e,s_1,s_2} p\left(B,e\right)\left(\theta - s_1 + A\right) + \left(1 - p\left(B,e\right)\right)\left(-s_2 + A\right) - c\left(e\right)$$

s.t.

$$E\pi_l = p\left(B,e\right)s_1 + \left(1 - p\left(B,e\right)\right)s_2 - B \geq 0 \tag{PC}$$

$$\theta - s_1 + A \geq 0 \tag{LL1}$$

$$-s_2 + A \geq 0 \tag{LL2}$$

where $(s_1, s_2)$ is the repayment from the firm to the lender if the project succeeds or fails, respectively.

**Lemma 1** $E\pi_l^* = 0$

Lemma 1 shows that, in equilibrium, the financier's participation constraint always binds due to Bertrand competition. The equilibrium borrowing amount $B^*\left(e_j\right)$ depends on effort $e$ and it satisfies

$$p'_B\left(B^*\left(e_j\right), e_j\right) = \frac{1}{\theta}, \quad j = H, L$$

We further assume that it is optimal to exert high effort for all firms.

**Assumption 6** Exerting $e_H$ is optimal.

Assumption 6 implies that, in equilibrium, a firm's expected profit if exerting $e_H$ is strictly higher than the expected profit if exerting $e_L$ for any $A$:

$$p\left(B^*\left(e_H\right), e_H\right)\theta - B^*\left(e_H\right) + A - c_H > p\left(B^*\left(e_L\right), e_L\right)\theta - B^*\left(e_L\right) + A$$

Equivalently,

$$\theta > \frac{\left(B^*\left(e_H\right) - B^*\left(e_L\right)\right) + c_H}{p\left(B^*\left(e_H\right), e_H\right) - p\left(B^*\left(e_L\right), e_L\right)}$$

**Proposition 2** *The optimal contract $(B, s_1, s_2|A)$ is a state-contingent contract. In equilibrium,*

*(1) $B = B^*\left(e_H\right) = B^*$, where $p'_B\left(B^*, e_H\right) = \frac{1}{\theta}$*

*(2) Repayment schemes $\left(s_1^*, s_2^*|s_1 \leq \theta + A, s_2 \leq A, s_1^* = \frac{B^* - (1-p(B^*, e_H))s_2^*}{p(B^*, e_H)}\right)$ are not unique,*

Proposition 2 shows that the optimal borrowing amount $B^*$ increases as $\theta$ increases.

Due to the risk-neutrality of the firms and the financiers, the optimal repayment scheme is indeterminate. According to Modigliani and Miller's (1958) theorem on the irrelevance of firms' financial structure, any equilibrium repayment scheme that satisfies Proposition 2, is optimal. Therefore, standard debt, equity and convertible debt contracts are all optimal in the benchmark case. Note that in this paper, we assume that the costs of signing different types of contracts are the same. Without loss of generality[4], we assume that the cost is equal to zero. If the costs were different, the contract with the lowest cost would be optimal.

**Assumption 7** The cost of signing standard debt, equity and convertible debt contracts equals to zero.

In the following sections, we analyze these three types of contracts which are commonly used in reality –standard debt, equity and convertible debt contracts– and show that if the firm returns and effort are observable and verifiable, all three types of contracts are optimal.

### 2.1.1 Standard debt contract

A standard debt contracts $(B, r|A)$ specifies the firm's borrowing amount $B$ and the corresponding interest rate $r$ for a firm with initial asset $A$. In equilibrium, the contract $(B, r|A)$ satisfies the financier's participation constraint (PC),

$$p\left(B, e\right)Br + \left(1 - p\left(B, e\right)\right)\min\left(Br, A\right) - B \geq 0 \tag{PC}$$

[4] As long as the costs of signing different types of contracts are equal, there is no difference between assuming cost = 0 or cost = c (>0), where c is constant.

the firms' limited liability constraints (LL1&LL2),

$$\theta - Br + A \geq 0 \tag{LL1}$$

$$-\min(Br, A) + A \geq 0 \tag{LL2}$$

and the firm's expected profit is maximized.

**Proposition 3** *The equilibrium standard debt contract* $(B, r|A)$ *is optimal for all risk-neutral firms, where in equilibrium*

*(1)* $B = B^*$

*(2)* $r = 1$ *for* $A \geq B^*$*; and* $r = \frac{B^* - (1-p(B^*, e_H))A}{p(B^*, e_H)B^*} > 1$ *for* $A < B^*$

For unconstrained firms (firms with initial assets $A \geq B^*$), in equilibrium, both limited liability constraints do not bind. Besides, they have enough initial assets to repay fully even if the project fails. Their financiers always obtain the full repayment, thus, their debt is secured, and the equilibrium interest rate for the unconstrained firms is equal to 1.

For constrained firms (firms with initial assets $A < B^*$), one limited liability constraint LL2 binds in equilibrium. This means that if the project fails, the firms go bankrupt, and their assets are liquidated by the financiers. The financiers cannot receive the full repayment once the project fails, thus, the equilibrium interest rate is higher than 1 in order to satisfy the financiers' participation constraint. Besides, the equilibrium interest rate for constrained firms decreases when the firm's initial asset increases.

Given that the firm returns are observable and verifiable by a third party, under standard debt contracts, firms do not default strategically (i.e., they do not default when the project succeeds). If the project succeeds, once the firms default, their assets will be liquidated and the financiers will still obtain full repayment as if the firms do not default because the firm returns will be verified and thus they have to repay fully.

### 2.1.2 Equity contract

An equity contract $(B, s|A)$ specifies the investment amount $B$ that the financier invests and the share $s$ $(0 < s < 1)$ of the firm('s value) that the financier obtains (at the end of the period). Specifically, the profit the financier obtains is $s\theta$ if the project succeeds, and $sA$ if the project fails. In equilibrium, the contract $(B, s|A)$ for a given $A$ satisfies the financiers' participation constraint (PC),

$$p(B, e)\, s\theta + (1 - p(B, e))\, sA - B \geq 0 \tag{PC}$$

the firms' limited liability constraints (LL1&LL2)

$$\theta - s\theta + A \geq 0 \tag{LL1}$$

$$sA + A \geq 0 \tag{LL2}$$

and the firms' expected profits are maximized.

**Proposition 4** *The equilibrium equity contract* $(B, s|A)$ *given an initial asset* $A$ *is optimal, where in equilibrium*

*(1)* $B = B^*$

*(2)* $s = \frac{B^*}{p(B^*, e_H)\theta + (1 - p(B^*, e_H))A}$

Since the financier obtains his share of the firm's profit automatically, the financier's expected profit is the same whether the firm returns are verifiable or not.

### 2.1.3 Convertible debt contract

**Definition 5** *Convertible debt contracts*

*A convertible debt contract is a standard debt contract plus a call option which gives the financier a unilateral right to convert the debt to equity at a predetermined time and with a predetermined conversion rate.*

A convertible debt contract $(B, r, \alpha|A)$ for a firm with initial asset $A$ specifies the borrowing amount $B$ and interest rate $r$ and a predetermined conversion rate $\alpha$ $(0 < \alpha < 1)$. Specifically, the financier receives $\alpha y$ if the financier converts debt to equity. The equilibrium convertible debt contract maximizes the firm's expected profit subject to the financier's participation constraint (PC)

$$p(B, e) \max(Br, \alpha\theta) + (1 - p(B, e)) \max(\min(A, Br), \alpha A) - B \geq 0 \tag{PC}$$

and the two limited liability constraints of the firm (LL1&LL2)

$$\theta - \max(Br, \alpha\theta) + A \geq 0 \tag{LL1}$$

$$-\max(\min(A, Br), \alpha A) + A \geq 0 \tag{LL2}$$

In equilibrium, if the project succeeds, firms have no incentive to default strategically because if they do so, the financiers can either simply convert the debt to equity and thus share the profit, or even if they do not convert, they can go to the court and verify the firms' return. Hence, the financiers can obtain the same repayment whether they convert or not, and the firms will not default strategically. If the project fails, in equilibrium, the constrained firms always default, and the financiers will not convert because $\alpha A < A$. As for the unconstrained firms, they have no incentives to default, and the financiers obtain the full repayment.

**Proposition 6** *The equilibrium convertible debt contract* $(B, r, \alpha|A)$ *given an initial assets* $A$ *is optimal, where*

*(1)* $B = B^*$ *for all firms*

*(2)* $r = 1$ *for* $A \geq B^*$*; and* $r = \frac{B^*-(1-p(B^*,e_H))A}{p(B^*,e_H)B^*}$ *for* $A < B^*$

*(3)* $\alpha = \frac{B^*}{\theta}$ *for* $A \geq B^*$*; and* $\alpha = \frac{B^*-(1-p(B^*,e_H))A}{p(B^*,e_H)\theta}$ *for* $A < B^*$

In the previous section, under verifiability of firm returns and risk-neutrality of both firms and their financiers, all three types of contracts: standard debt, equity and convertible debt contracts are optimal. Since under verifiability for all three types of contracts, firms have no incentives to default strategically. Besides, under risk-neutrality of both firms and their financiers, firms are indifferent among all three contracts since in equilibrium firms' expected profits are the same and are maximized under all three types of contracts.

Through the following sections in this paper, we assume that firm returns are observable to both the firms themselves and their financiers, but not verifiable by a third party (Assumption 3), which is crucial to our analysis.

## 2.2 Non-verifiable firm returns

Now, suppose that firms' project returns are not verifiable by a third party (Assumption 3). For unconstrained firms ($A \geq B^*$), all three types of contracts are optimal. The reason is that under standard debt contracts, unconstrained firms do not have incentives to default strategically. Unconstrained firms have enough assets such that even if they default, the financiers still obtain the full repayment $B^*$. Moreover, since convertible debt and equity contracts are both immune to ex-post strategic default, they are optimal under non-verifiability of firm returns.

For constrained firms, standard debt contracts are not optimal. The reason is that constrained firms have incentives to default strategically under standard debt contracts. Because of non-verifiability of firm returns and limited liability of the firms, if the project succeeds, a constrained firm's profit is $\theta$ if it defaults and its profit is $\theta - \frac{B^*-A}{p(B^*,e_H)}$ if it does not default. The firm's profit is higher if it defaults.

$$\underbrace{\theta}_{\text{default}} > \underbrace{\theta - \frac{B^* - A}{p\left(B^*, e_H\right)}}_{\text{not default}}$$

As a result, equilibrium standard debt contracts must mitigate the firms' incentives to default strategically. That is, if the project succeeds, the equilibrium repayment must not exceed the firms' initial assets $A$. Therefore, the equilibrium borrowing amount $B$ equals the firms' initial assets and standard debt contracts are not optimal for constrained firms.

Equity contracts and convertible debt contracts are both immune to ex-post strategic default due to non-verifiability of firm returns. An equity holder shares the firm's return automatically. Convertible debt contracts grant the financiers a unilateral right to convert debt to equity and thus to share firms' return if the project succeeds.

**Proposition 7** *Under non-verifiability of firm returns,*

*(1) For unconstrained firms, standard debt, equity and convertible debt are all optimal*

*(2) For constrained firms, standard debt contracts are dominated. Equity and convertible debt contracts are optimal*

So far, we have shown that under non-verifiability of firm returns, standard debt contracts are dominated by the other two types of contracts. However, it is not enough to show the optimality of convertible debt contracts for smaller firms. Nor is it enough to explain the fact that the probability of using convertible debt contracts is positively related to the firms' profit-to-assets ratios. In the following two sections, we only consider the environment with non-verifiable firm returns (under Assumption 3) through the whole following paper. Besides, we further relax Assumption 2 and then Assumption 1 each by each.

# 3 Generalized Model (1): Moral hazard

In this section, we generalize the model and relax the assumption of observable firm effort (Assumption 2). The moral hazard problem is generated by the dependence of the distribution of the project returns on the firms' effort choice, which is unobservable to the financiers. In order to provide incentives to the firms to exert high effort, it is necessary to let the firms bear some risk of the project. In the following analysis, we again focus on the three types of contracts and derive the equilibrium contracts under moral hazard.

## 3.1 Standard debt contract

The equilibrium standard debt contracts for unconstrained firms are the solution to the following problem:

$$\max_{B,e,r} p(B,e)(\theta - Br + A) + (1 - p(B,e))(-Br + A) - c(e)$$

s.t.

$$E\pi_l = p(B,e)\,Br + (1 - p(B,e))\,Br - B \geq 0 \tag{PC}$$

$$\theta - Br + A \geq 0 \tag{LL1}$$

$$-Br + A \geq 0 \tag{LL2}$$

$$(p(B,e_H) - p(B,e_L))\,\theta \geq c_H \tag{IC}$$

**Proposition 8** *For $A \geq B^*$, the equilibrium standard debt contract $(B, r|A)$ is optimal,*
*(1) $B = B^*$*
*(2) $r = 1$*

The equilibrium standard debt contracts for unconstrained firms are exactly the same as the ones in Proposition 3. The unconstrained firms pay a fixed repayment $B^*$ and keep the rest of the returns. Therefore, they have incentives to exert high effort, and the (IC) constraints do not bind.

As for constrained firms, as we mentioned before, the standard debt contracts in Proposition 3 will not be offered in equilibrium since the constrained firms have incentive to default strategically due to non-verifiability of firm returns. As a result, in equilibrium, standard debt contracts are clearly dominated by equity and convertible debt contracts.

## 3.2 Equity contract

If equity contracts are offered in equilibrium, the equilibrium equity contracts must solve the following problem:

$$\max_{B,e,s} p(B,e)(\theta - s\theta + A) + (1 - p(B,e))(-sA + A) - c(e)$$

s.t.

$$p(B,e)\,s\theta + (1 - p(B,e))\,sA - B \geq 0 \tag{PC}$$

$$\theta - s\theta + A \geq 0 \tag{LL1}$$

$$sA + A \geq 0 \tag{LL2}$$

$$(p(B,e_H) - p(B,e_L))(\theta - s(\theta - A)) \geq c_H \tag{IC}$$

**Proposition 9** *For $A \geq A_2$, the equilibrium equity contracts $(B, s|A)$ is optimal,*

*(1) $B = B^*$*

*(2) $s = \frac{B^*}{p(B^*,e_H)\theta+(1-p(B^*,e_H))A}$*

*where*

$$A_2 = \frac{\theta\left(B^* - \left(\frac{c_H}{p(B^*,e_H)-p(B^*,e_L)}\right) p\left(B^*, e_H\right)\right)}{B^* + \left(\frac{c_H}{p(B^*,e_H)-p(B^*,e_L)}\right)\left(1 - p\left(B^*, e_H\right)\right)}$$

For $A \geq A_2$, the equilibrium equity contracts are the same as the ones stated in Proposition 4. This shows that for the firms with higher $A\,(A \geq A_2)$, equity contracts are optimal. However, for the firms with smaller $A\,(A < A_2)$, there will be no equity contracts offered in equilibrium due to the (IC) constraints.

## 3.3 Convertible debt contract

The equilibrium convertible debt contracts are the solution to the following problem:

$$\max_{B,e,\alpha} p\left(B, e\right)\left(\theta - \max\left(Br, \alpha\theta\right) + A\right) + \left(1 - p\left(B, e\right)\right)\left(0 - \max\left(\min\left(A, Br\right), \alpha A\right)\right) - c\left(e\right)$$

s.t.

$$p\left(B, e\right)\max\left(Br, \alpha\theta\right) + \left(1 - p\left(B, e\right)\right)\max\left(\min\left(A, Br\right), \alpha A\right) - B \geq 0 \tag{PC}$$

$$\theta - \max\left(Br, \alpha\theta\right) + A \geq 0 \tag{LL1}$$

$$-\max\left(\min\left(A, Br\right), \alpha A\right) + A \geq 0 \tag{LL2}$$

$$\left(p\left(B, e_H\right) - p\left(B, e_L\right)\right)\left[\left(\theta - \max\left(Br, \alpha\theta\right) + A\right) + \max\left(\min\left(A, Br\right), \alpha A\right)\right] \geq c_H$$

**Proposition 10** *For $A \geq A_1$, the equilibrium convertible debt contract $(B, r, \alpha|A)$ is optimal,*

*(1) $B = B^*$ for all firms*

*(2) $r = 1$ for $A \geq B^*$; and $r = \frac{B^*-(1-p(B^*,e_H))A}{p(B^*,e_H)B^*}$ for $A_1 \leq A < B^*$*

*(3) $\alpha = \frac{B^*}{\theta}$ for $A \geq B^*$; and $\alpha = \frac{B^*-(1-p(B^*,e_H))A}{p(B^*,e_H)\theta}$ for $A_1 \leq A < B^*$*

*where*

$$A_1 = B^* - p\left(B^*, e_H\right)\left(\theta - \frac{c_H}{p\left(B^*, e_H\right) - p\left(B^*, e_L\right)}\right)$$

Proposition 10 demonstrates that convertible debt contracts are optimal for any $A \geq A_1$ if firms' effort is unobservable.

We show that, for unconstrained firms, all three types of contracts are still optimal even if taking into account the moral hazard problem since the (IC) constraints do not bind for larger firms. For constrained firms, standard debt contracts are not optimal because the firms' have incentives to default strategically. Equity contracts and convertible debt contracts both solve this ex-post strategical default. However, here due to the moral hazard problem, equity contracts are not optimal for firms with initial assets $A$ smaller than a threshold $A_2$. For a firm with small initial asset $A < A_2$, in order to induce it to exert high effort, the share $s$ assigned to the financier can not be too high. However, if the share is not high enough, the financier's participation constraint (PC) is violated (i.e., financier has negative profits) because the share $s$ is too low. As a result, in equilibrium, there is no equity contract offered to the firms with $A < A_2$. This means that there exist a trade-off between ex-post and ex-ante efficiency. In order to prevent ex-post inefficiency caused by unobservable effort, the ex-ante efficiency would be sacrificed, that is, small firms will not be able to obtain the funds to invest in projects with positive present value. Convertible debt contracts result in the same trade-off as equity contracts due to moral hazard. However, the ex-ante inefficiency is a less severe problem in convertible debt contracts than in equity contracts. Because in equilibrium, $\alpha < s$, there exists another threshold $A_1$ such that for a firms with $A_1 \leq A < A_2$, convertible debt contracts are optimal. We show $A_1 < A_2$ in the following lemma.

**Lemma 11** $A_1 < A_2$

In summary, unconstrained firms have no incentives to default strategically ex-post even though the firm returns are not verifiable. Besides, unconstrained firms are not constrained by the incentive compatibility constraints even though there is a moral hazard problem due to unobservable firm effort. Therefore, for unconstrained firms, standard debt, equity and convertible debt contracts are all optimal. However, this result does not hold for the constrained firms. First of all, constrained firms always have incentives to default strategically if the firm returns are not verifiable. The solutions to this ex-post strategical default are either to let the financier be an equity holder or a convertible debt holder. In other words, both equity and convertible debt contracts can effectively solve the ex-post strategical default of the firm returns are not verifiable. However, equity contracts are dominated by convertible debt contracts for firms with initial assets $A_1 \leq A < A_2$ because there is no equity contract for a firm with $A_1 \leq A < A_2$ such that the incentive compatibility constraint of the firm and the financier's participation constraint are both satisfied at the same time. Therefore, while equity and convertible debt contracts are both optimal for constrained firms with $A \geq A_2$, for constrained firms with $A_1 \leq A < A_2$, only convertible debt contracts are optimal.

This result is consistent with the stylized facts that the probability of using convertible debt is negatively related with the firm size (Lewis, Rogalski and Seward (1999)). In particular, for small firms ($A_1 < A < A_2$), only convertible debt contracts are optimal.

Moreover, under moral hazard, we are able to explain another stylized fact (Lewis, Rogalski and Seward (1999)) – the positive relation between firms' profit-to-asset ratio and the probability of using convertible debt – in empirical evidence. Firms' profit-to-asset ratio is defined as

$$\text{profit-to-asset ratio } = \frac{p\left(B^*, e_H\right)\theta - B^* + A}{A}$$

If $\theta$ increases, $B^*$ increases (see Proposition 2). The profit-to-asset ratio also increases,

$$\begin{aligned} \frac{\partial\left(\frac{p(B^*,e_H)\theta - B^* + A}{A}\right)}{\partial\theta} &= \frac{p\left(B^*, e_H\right)}{A} + \frac{p'_B\left(B^*, e_H\right)\theta - 1}{A}\frac{\partial B^*}{\partial\theta} \\ &= \frac{p\left(B^*, e_H\right)}{A} > 0 \end{aligned} \tag{1}$$

Moreover, if $\theta$ increases, the threshold $A_1$ decreases and $A_2$ increases.

**Lemma 12** $\frac{\partial A_1}{\partial\theta} < 0$

**Lemma 13** $\frac{\partial A_2}{\partial\theta} > 0$

Note that the change of the function form of $p\left(B, e\right)$ also affects the firms' profit-to-asset ratio. We further derive the relation between the threshold $A_1$, $A_2$ and the probability function of success $p\left(B, e\right)$ in the following lemma.

**Lemma 14** *Suppose* $p\left(B, e_j\right)$ *is a homogeneous function with degree* $n$ *(*$n < 1$*). Let* $F\left(p\left(B, e_j\right)\right) = \widetilde{p}\left(B, e_j\right)$ *be a homothetic function, where* $F\left(p\left(B, e_j\right)\right)$ *is monotone increasing in* $p\left(B, e_j\right)$*.* $A_1$ *under* $p\left(B^*, e_j\right)$ *is higher than* $\widetilde{A_1}$ *derived under* $\widetilde{p}\left(\widetilde{B}^*, \widetilde{e_j}\right)$*, where* $j = H, L$

**Lemma 15** *Suppose* $p\left(B, e_j\right)$ *is a homogeneous function with degree* $n$ *(*$n < 1$*). Let* $F\left(p\left(B, e_j\right)\right) = \widetilde{p}\left(B, e_j\right)$ *be a homothetic function, where* $F\left(p\left(B, e_j\right)\right)$ *is monotone increasing in* $p\left(B, e_j\right)$*.* $A_2$ *under* $p\left(B^*, e_j\right)$ *is lower than* $\widetilde{A_2}$ *derived under* $\widetilde{p}\left(\widetilde{B}^*, \widetilde{e_j}\right)$*, where* $j = H, L$

If the distribution of firms' initial assets is further assumed to be exogenous, we derive that the probability of a firm being constrained by limited liability increases as the firm's profit-to-asset ratio increases. Using the result from Lemma 12, Lemma 13, Lemma 14, Lemma 15 and equation (1) derived above, we show that the probability of a firm using convertible debt also increases as the probability of the firm being constrained increases.

**Assumption 8** The distribution of firms' initial assets is exogenous

**Proposition 16** *If the firms' effort is unobservable, the probability of a firm using a convertible debt contract increases as the firm's profit-to-asset ratio increases.*

The result of Proposition 16 comes directly from Lemma 12, Lemma 13, Lemma 14, Lemma 15 and equation (1). Besides, we have shown that for firms with initial assets $A \in [A_1, A_2]$, convertible debt contracts are optimal. The probability of a firm using convertible debt can be written as

$$prob\,(A \in [A_1, A_2])$$

First, from the results of Lemma 12, Lemma 13, and equation (1), we can conclude that the probability of a firm using convertible debt contracts increase as the firms' profit-to-assets ratio increases which is due to an increase in $\theta$. The reason is that as $\theta$ increases, $A_1$ decreases and $A_2$ increases.

Second, if the firms' profit-to-assets ratios increase is due to an increase in the function $p\,(B, e_j)$ $(j = H, L)$, from Lemma 14, Lemma 15, we conclude that $A_1$ decreases to $\widetilde{A}_1$ and $A_2$ increases to $\widetilde{A}_2$. Therefore,

$$prob(A \in [\widetilde{A}_1, \widetilde{A}_2]) > prob\,(A \in [A_1, A_2])$$

# 4 Generalized model (2): Risk aversion

In this section, we assume Assumption 2 holds and we generalize the model by relaxing Assumption 1. We consider risk averse firms and risk neutral financiers.

To compare the three types of contracts under the assumption of the firms being risk averse is more complicated. Note that the probability of success $p\,(B, e)$ is endogenous. In particular, the probability of success can be increased through a higher borrowing amount $B$ and a higher effort $e$ of firms. Under risk aversion of firms, different types of contracts may result in different equilibrium borrowing $B$ as well as different $e$. This in turns affect the probability of success $p\,(B, e)$. Hence, at this stage it is not clear to determine which type of contracts is optimal under risk aversion of the firms without further assumptions.

In order to analyze this problem, we introduce the concept of " mixed risk aversion", which is defined in Caballé and Pomansky (1996). They consider that the distribution function of outcomes is endogenous and it can be influenced by agents' behavior. This is the concept so called self-protection (Ehrlich and Becker (1972)). Caballé and Pomansky (1996) show that if the firms' utility functions satisfy $(-1)^{n+1} U^{(n)} \geq 0$, the measurement of mixed risk aversion is monotonic with Arrow-Pratt risk aversion. In other words, if an agent is more risk averse than the other, we can also conclude that the agent is more mixed risk averse than the other. Moreover, they provide a comparative study which allows us to analyze our problem and to compare the three types of contracts.

Suppose equity contracts are offered in equilibrium, the equilibrium equity contracts solve the following problem

$$\max_{B,e,s} p\left(B,e\right) U\left(\theta - s\theta + A\right) + \left(1 - p\left(B,e\right)\right) U\left(-sA + A\right) - C\left(e\right)$$

s.t.

$$p\left(B,e\right) s\theta + \left(1 - p\left(B,e\right)\right) sA - B \geq 0$$

$$0 < s \leq 1$$

We denote the equilibrium equity contracts $\left(B^E, s^E|A\right)$ where $B^E$ is the equilibrium borrowing amount and $s$ is the equilibrium share of the firms' profits promised to the financiers. Since effort is observable, equilibrium equity contracts depend on the firms' effort choice. We further assume that it is optimal for firms to exert high effort $e_H$ even under risk aversion.

In the previous analysis, we have shown that standard debt contracts are dominated by convertible debt contracts if the firms are constrained by limited liability. As for unconstrained firms, standard debt and convertible debt contracts are both optimal and they achieve the same equilibrium contract outcomes. Hence, In the following, we only need to compare convertible debt contracts with equity contracts.

Suppose convertible debt contracts are offered in equilibrium, they solve the following problem:

$$\max_{B,e,\alpha} p\left(B,e\right) U\left(\theta - \alpha\theta + A\right) + \left(1 - p\left(B,e\right)\right) U\left(0 - \left(\min\left(-Br + A, \alpha A\right), 0\right)\right) - C\left(e\right)$$

s.t.

$$p\left(B,e\right) \alpha\theta + \left(1 - p\left(B,e\right)\right) \min\left(-Br + A, \alpha A\right) - B \geq 0$$

$\left(B^{CD}, r^{CD}, \alpha^{CD}|A\right)$ denotes the equilibrium convertible debt contracts.

Equity contracts make the financiers bear more risk compared to convertible debt contracts. In particular, the difference of the firms' utility between good outcome (the project succeeds) and bad outcome (the project fails) is smaller under equity contracts. This in turns leads to the following result.

**Lemma 17** $B^E < B^{CD}$

Lemma 17 demonstrates that under convertible debt contracts, risk-averse firms will choose a higher borrowing amount $B^{CD}$, which implies a higher probability of success of the project under convertible debt. The intuition is the following: the utility of a firm at bankruptcy is lower under convertible debt contracts than the utility under equity contracts given the same loan size. Therefore, due to the firm's risk aversion, the firm will choose a higher equilibrium loan size under convertible debt in order to decrease the probability of bankruptcy.

Caballé and Pomansky (2000) and Dachraoui et al. (2000) show that more mixed risk averse individuals choose higher self-protection or are more willing to pay more for lowering the probability of the bad outcome when this probability is low. Although they focus on comparing agents' with different degrees of risk aversion given the same type of contracts, their results shed some light on our result of Proposition 18

**Proposition 18** *If the firms are mixed risk averse, and if* $p\left(B^E, e_H\right) > \frac{1}{2}$*(i.e,* $1 - p\left(B^*, e_H\right) < \frac{1}{2}$*), equity contracts are dominated by convertible debt contracts and thus, convertible debt contracts are optimal.*

The intuition of Proposition 18 is that if the probability of failure of the project is already low even under equity financing, which implies that the probability of failure is very low, this state (failure of the project) can be negligible. As a result, firms are better off if choosing convertible debt contracts, since under convertible debt contracts, the borrowing amount $B^{CD}$ is higher than $B^E$, and thus the probability of success is also higher $\left(p\left(B^{CD}, e_H\right) > p\left(B^E, e_H\right)\right)$. Therefore, the firms' utility if the project succeeds is higher under convertible debt than under equity contracts.

# 5 Discussion

In the previous sections, we have generalized the model in the two directions given non-verifiability of firm returns each by each. In reality, it is plausible that both moral hazard problem and risk aversion exist at the same time.

From the analysis in Generalized Model (1), we have shown that for larger firms, in particular, for $A \geq B^*$, the firms' (IC) constraints do not bind in equilibrium. Therefore, under the assumption of both firms and their financiers being risk-neutral, all three types of contracts are optimal. As for firms with assets $A_2 \leq A < B^*$, both convertible debt and equity contracts are optimal. Standard debt contracts are dominated because the firms' incentive of strategical default. Finally, for the small firms with initial assets $A_1 \leq A < A_2$, only convertible debt contracts are optimal.

In the analysis in Generalized Model (2), we have shown that under some conditions, constrained firms prefer convertible debt contracts over other types of contracts (Proposition 18). For unconstrained firms, both standard debt and convertible debt contracts are optimal.

Combing both results from the analyses, if we consider an environment in which firms are risk averse and the firms' effort is unobservable, we conjecture that for unconstrained firms, both convertible debt and standard debt contracts are optimal, and equity contracts are dominated due to risk aversion of the firms. For constrained firms, standard debt contract are dominated because the constrained firms have incentives to default strategically due to non-verifiability firm returns. As a result, only convertible debt contracts are optimal since convertible debt on the one hand, induces the constrained firms to exert high effort, and on the other hand, achieves better risk sharing compared to equity contracts.

# 6 Conclusion

In this paper, we analyze optimal financial contracts of firms with heterogeneous initial assets. We start at building a simple model with both risk-neutral firms and financiers and we show that under non-verifiability of firm returns, small firms have incentives to default strategically under standard debt contracts. Equity and convertible debt contracts can prevent the ex-post strategical default.

Further, we generalize the model and consider two additional dimensions each by each: (1) moral hazard caused by firms' unobservable effort, and (2) risk aversion of firms or/and their financiers. In (1), only convertible debt contracts are optimal for small firms. Standard debt contracts and equity contracts are dominated because of ex-post strategical default and moral hazard, respectively. In (2), firms are risk averse and their financiers are risk neutral. Under some condition, in particular, if the probability of failure of the project is lower than $\frac{1}{2}$ under equity contracts, risk -averse firms will be better off if using convertible debt contracts since under this condition, firms prefer to choose higher borrowing amount $B$ and thus attain a higher probability of success of the project. This argument is true for all firms if they all have the same level of risk aversion. Since for unconstrained firms, both standard debt contracts and convertible debt contracts achieve the same contract outcomes in equilibrium, hence, both standard debt and convertible debt contracts are optimal for unconstrained firms.

In sum, when we consider both (1) and (2) together, large firms, even under non-verifiability of firm returns, do not have incentives to default strategically. Moreover, large firms have higher initial assets, thus they have incentives to exert high effort. However, due to risk aversion, larger firms prefer standard debt and convertible debt contracts over equity contracts. On the other hand, small firms have incentives to default under non-verifiability of firm returns. Standard debt contracts are dominated by convertible debt and equity contracts. Moreover, due to the risk aversion of the firms, convertible debt contracts dominate equity contracts since convertible debt not only induces the constrained firms to exert high effort but only achieves better risk sharing. Therefore, the optimality of convertible debt for small firms is proved.

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# 8 Appendix

**Proof. Lemma 1**

Suppose that $\left(\overline{B}, \overline{s_1}, \overline{s_2}|A\right)$ is an equilibrium contract and it yields a positive expected profit to the lender,

$$E\pi_l\left(\overline{B}, \overline{s_1}, \overline{s_2}|A\right) > 0$$

The other financier can offer another contract $(B', s_1', s_2'|A)$ where $B' = \overline{B}$, $s_1' = \overline{s_1} - \varepsilon$ and $s_2' = \overline{s_2} - \varepsilon$ and the lender still have non-negative expected profit:

$$E\pi_l\left(\overline{B}, \overline{s_1}, \overline{s_2}|A\right) > E\pi_l\left(B', s_1', s_2'|A\right) > 0$$

This contract $(B', s_1', s_2'|A)$ gives the firm higher expected return. Hence, $\left(\overline{B}, \overline{s_1}, \overline{s_2}|A\right)$ is not an equilibrium contract. By doing so, the equilibrium contract should satisfy $E\pi_l^* = 0$. ■

**Proof. Proposition 2**

The optimal contract solves the following problem

$$\max_{B,e} p(B,e)(\theta - s_1 + A) + (1 - p(B,e))(-s_2 + A)$$

s.t.

$$E\pi_l = p(B,e)s_1 + (1 - p(B,e))s_2 - B \geq 0 \tag{PC}$$

$$\theta - s_1 + A \geq 0 \tag{LL1}$$

$$-s_2 + A \geq 0 \tag{LL2}$$

First, we ignore the two (LL) constraints. In equilibrium, $E\pi_l = 0$. Hence,

$$p(B,e)(s_1 - s_2) + s_2 = B$$

Plugging this into the firm's objective function. For $(s_1, s_2)$ satisfying (LL) and (IC), $B$ and $e$ solve

$$\max_{B,e} p(B,e)\theta - (p(B,e)(s_1 - s_2) + s_2) + A \equiv \max_{B,e} p(B,e)\theta - B + A$$

Under Assumption 6, in equilibrium, $e = e_H$. Therefore, the optimal borrowing amount

$$B^*(e_H) = B^*$$

and it satisfies

$$p_B'(B^*, e_H) = \frac{1}{\theta}$$

Since both firms and their financiers are risk-neutral, the optimal repayment scheme $(s_1^*, s_2^*)$ is indetermined and it satisfies the following relation

$$s_1^* = \frac{B^* - (1 - p(B^*, e_H))\, s_2^*}{p(B^*, e_H)}$$

■

**Proof. Proposition 3**(*Standard debt contracts*)

The equilibrium standard debt contract is the solution to the following problem

$$\max_{B,e} p(B,e)(\theta - Br + A) + (1 - p(B,e))(-Br + A)$$

s.t.

$$p(B,e)\, Br + (1 - p(B,e))\, Br - B \geq 0 \tag{PC}$$

$$\theta - Br + A \geq 0 \tag{LL1}$$

$$-Br + A \geq 0 \tag{LL2}$$

First, we ignore LL1 and LL2 and solves the problem. Under Assumption 6, $e = e_H$, and thus we derive the equilibrium $B = B^*(e_H) = B^*$.

For unconstrained firms ($A \geq B^*$), both LL1 and LL2 do not bind. Equilibrium $r = 1$.

For constrained firms ($A < B^*$), LL2 binds. Hence, in equilibrium,

$$r = \frac{B^* - (1 - p(B^*, e_H))\, A}{p(B^*, e_H)\, B^*} > 1$$

All firms' expected profits are maximized. ■

**Proof. Proposition 4** (*Equity contracts*)

The equilibrium equity contract is the solution to the following problem

$$\max_{B,e} p(B,e)(\theta - s\theta + A) + (1 - p(B,e))(-sA + A)$$

s.t

$$p(B,e)\, s\theta + (1 - p(B,e))\, sA - B \geq 0 \tag{PC}$$

$$\theta - s\theta + A \geq 0 \tag{LL1}$$

$$-sA + A \geq 0 \tag{LL2}$$

From Lemma 1, (PC) binds in equilibrium. Besides, due to Assumption 6, $e = e_H$, and thus, equilibrium $B = B^*$ where $p'_B(B^*, e_H) = \frac{1}{\theta}$. And the equilibrium share

$$s = \frac{B^*}{p(B^*, e_H)\,\theta + (1 - p(B^*, e_H))\, A}$$

Moreover, in equilibrium, for unconstrained firms, $s\theta \geq B^* > sA$ always holds if $\theta > \overline{A}$. We prove this result by contradiction:

(a) suppose $s\theta > sA > B^*$, in equilibrium, $E\pi_l = 0$. Hence, equilibrium

$$B = B^*$$

and

$$s = \frac{B^*}{p(B^*, e_H)\,\theta + (1 - p(B^*, e_H))\,A}$$

Since $s\theta > sA > B^*$, we have

$$\frac{B^*\theta}{p(B^*, e_H)\,\theta + (1 - p(B^*, e_H))\,A} > \frac{B^*A}{p(B^*, e_H)\,\theta + (1 - p(B^*, e_H))\,A} > B^*$$

This is equivalent to

$$\frac{\theta}{p(B^*, e_H)\,\theta + (1 - p(B^*, e_H))\,A} > \frac{A}{p(B^*, e_H)\,\theta + (1 - p(B^*, e_H))\,A} > 1$$

From second part of inequality, we have

$$\frac{A}{p(B^*, e_H)\,\theta + (1 - p(B^*, e_H))\,A} > 1$$

Hence,

$$p(B^*, e_H)\,(\theta - A) < 0$$

which contradicts with $\theta > A$

(b) suppose $B^* > s\theta > sA$, the same argument as above, in equilibrium,

$$B = B^*$$

and hence

$$s = \frac{B^*}{p(B^*, e_H)\,\theta + (1 - p(B^*, e_H))\,A} > 1$$

which is impossible.

Therefore, the equilibrium $B^*$ and $s$ must satisfy $s\theta > B^* > sA$ if $\theta > \overline{A}$. On the other hand, if $\theta \leq \overline{A}$, for $\theta \leq A \leq \overline{A}$, $s\theta < B^* \leq sA$ must hold in equilibrium. ■

**Proof. Proposition 6** (*Convertible debt contracts*)

For unconstrained firms, the equilibrium convertible contract solves the following problem

$$\max_{B,e}\; p(B, e)\,(\theta - \max(\alpha\theta, Br) + A) + (1 - p(B, e))\,(-\max(\alpha A, Br) + A)$$

s.t.

$$p(B,e)\max(\alpha\theta, Br) + (1 - p(B,e))\max(\alpha A, Br) - B \geq 0 \tag{PC}$$

$$\theta - \max(\alpha\theta, Br) + A \geq 0 \tag{LL1}$$

$$-\max(\alpha A, Br) + A \geq 0 \tag{LL2}$$

In equilibrium, under Assumption 6, $e = e_H$ and thus, $B = B^*$. Hence, $r = 1$ for unconstrained firms ($A \geq B^*$). From the proof of Proposition 5, it is shown that equilibrium $B^*$ must satisfy $\alpha\theta \geq B^* > \alpha A$ if $\theta > \overline{A}$. Therefore, if $\theta > \overline{A}$, the financier does not convert if the project fails, and is indifferent between converting or not converting if the project succeeds. In equilibrium, $E\pi_l = 0$ (Lemma 1). Thus, for unconstrained firms, equilibrium $\alpha = \frac{B^*}{\theta}$ if $\theta > \overline{A}$. If $\theta \leq \overline{A}$, $\alpha\theta < B^* \leq \alpha A$ must hold. Hence, the financier does not convert if the project succeeds, and is indifferent between converting and not converting if the project fails. Therefore, in equilibrium,

$$\alpha = \frac{B^*}{A}$$

For constrained firms, since $A < B^*$, $\alpha\theta \geq B^* > \alpha A$ always holds assuming $\theta > \overline{A}$. In equilibrium, the financier converts if the project succeeds and does not convert if the project fails and obtain the firm's total asset $A$. Therefore, in equilibrium,

$$\alpha = \frac{B^* - (1 - p(B^*, e_H))A}{p(B^*, e_H)\theta}$$

for constrained firms. ■

**Proof. Proposition 7** (*Non-verifiable firm returns*)

Equity and convertible debt contracts are immune to non-verifiability. Hence, in equilibrium, both contracts implement the optimal contracts derived in Proposition 2. However, standard debt contracts are not optimal for constrained firms because constrained firms have incentives to default strategically. Therefore, under standard debt contracts, in equilibrium, $B = A$, and the constrained firm's expected profits is $p(A, e_H)\theta$.

We can show that

$$p(B^*, e_H)\theta - B^* + A > p(A, e_H)\theta$$

if and only if

$$\theta > \frac{B^* - A}{p(B^*, e_H) - p(A, e_H)}$$

For any constrained firm ($A < B^*$), $\theta > \frac{B^* - A}{p(B^*, e_H) - p(A, e_H)}$ if (1)$\frac{B^* - A}{p(B^*, e_H) - p(A, e_H)}$ is decreasing in $A$ and (2) at $A = 0$, $p(B^*, e_H)\theta - B^* > 0$ holds

(2) is always true since we assume that the project is of net positive present value. And we show (1) also holds in the following:

$$\begin{aligned} & \frac{\partial \frac{B^*-A}{p(B^*,e_H)-p(A,e_H)}}{\partial A} \\ = \quad & \frac{p'(A,e_H)(B^*,e_H-A)-(p(B^*,e_H)-p(A,e_H))}{(p(B^*,e_H)-p(A,e_H))^2} \\ < \quad & 0 \end{aligned}$$

because

$$p'(A,e_H) > \frac{p(B^*,e_H)-p(A,e_H)}{B^*-A}$$

is always true for any $A < B^*$ since the function $p(.)$ is strictly concave. Therefore, (1) & (2) both hold and thus, $p(B^*,e_H)\theta - B^* + A > p(A,e_H)\theta$ is true for all $A < B^*$.

In conclusion, for $A < B^*$, standard debt contracts are dominated by equity and convertible debt contracts. ■

**Proof. Proposition** 8

The proof is the same as in Proposition 3. ■

**Proof. Proposition** 9

The proof follows Proposition 4 and (IC) bind for firms with $A < A_2$. ■

**Proof. Proposition** 10

The proof follows Proposition 6 and (IC) bind for firms with $A < A_1$. ■

**Proof. Lemma 11** (two initial asset thresholds)

$A_1$ is the threshold under convertible debt contracts such that

$$(p(B^*,e_H)-p(B^*,e_L))\left(\theta - \frac{B^*-A_1}{p(B^*,e_H)}\right) = c_H$$

Thus,

$$A_1 = B^* - p(B^*,e_H)\left(\theta - \frac{c_H}{p(B^*,e_H)-p(B^*,e_L)}\right)$$

And $A_2$ is the threshold under equity financing such that

$$(p(B^*,e_H)-p(B^*,e_L))(\theta - s(\theta - A_2)) = c_H$$

where

$$s = \frac{B^*}{p(B^*.e_H)\theta + (1-p(B^*,e_H))A}$$

Thus,

$$A_2 = \frac{\theta\left(B^* - \left(\frac{c_H}{p(B^*,e_H)-p(B^*,e_L)}\right)p(B^*,e_H)\right)}{B^* + \left(\frac{c_H}{p(B^*,e_H)-p(B^*,e_L)}\right)(1-p(B^*,e_H))}$$

Let

$$\theta - \frac{c_H}{p(B^*, e_H) - p(B^*, e_L)} = Q$$

We have

$$A_1 = B^* - p(B^*, e_H) Q$$

and

$$A_2 = \frac{\theta (B^* - p(B^*, e_H) Q)}{B^* + Q(1 - p(B^*, e_H))} = \frac{\theta A_1}{B^* + Q(1 - p(B^*, e_H))}$$

We can show that $A_2 > A_1$ if and only if

$$\frac{\theta A_1}{B^* + Q(1 - p(B^*, e_H))} > A_1$$

That is, we have to show $\theta > B^* + Q(1 - p(B^*, e_H))$.

$$\begin{aligned}
& \theta > B^* + (1 - p(B^*, e_H)) \left( \theta - \frac{c_H}{p(B^*, e_H) - p(B^*, e_L)} \right) \\
= \; & B^* + \theta - \frac{c_H}{p(B^*, e_H) - p(B^*, e_L)} - p(B^*, e_H) \left( \theta - \frac{c_H}{p(B^*, e_H) - p(B^*, e_L)} \right) \\
= \; & \theta + B^* - (1 - p(B^*, e_H)) \frac{c_H}{p(B^*, e_H) - p(B^*, e_L)} - p(B^*, e_H) \theta
\end{aligned}$$

Therefore, $A_2 > A_1$ as long as the following inequality holds.

$$\underbrace{p(B^*, e_H) \theta - B}_{+} + \underbrace{(1 - p(B^*, e_H)) \frac{c_H}{p(B^*, e_H) - p(B^*, e_L)}}_{+} > 0$$

Since by assumption, the project is of a positive net present value, $p(B^*, e_H) \theta - B$ is always positive. The inequality above holds and thus $A_2 > A_1$. ■

**Proof. Lemma** 12

We have

$$A_1 = B^* - p(B^*, e_H) \left( \theta - \frac{c_H}{p(B^*, e_H) - p(B^*, e_L)} \right)$$

and

$$\frac{\partial A_1}{\partial \theta} = -p(B^*, e_H) + \frac{\partial A_1}{\partial B^*} \frac{\partial B^*}{\partial \theta}$$

Since under

$$\frac{p'(B^*.e_H)}{p(B^*, e_H)} < \frac{p'(B^*, e_L)}{p(B^*, e_L)}$$

we have

$$\frac{\partial A_1}{\partial B^*} < 0$$

Besides,

$$\frac{\partial B^*}{\partial \theta} = -\frac{\frac{\partial^2 E\pi_f}{\partial B \partial \theta}}{\frac{\partial^2 E\pi_f}{\partial B^2}} > 0$$

As a result,

$$\begin{aligned} \frac{\partial A_1}{\partial \theta} &= -p\left(B^*, e_H\right) + \underbrace{\frac{\partial A_1}{\partial B^*}}_{-} \underbrace{\frac{\partial B^*}{\partial \theta}}_{+} \\ &< 0 \end{aligned}$$

■

**Proof. Lemma** 13

We have

$$A_2 = \frac{\theta\left(B^* - \left(\frac{c_H}{p(B^*,e_H)-p(B^*,e_L)}\right) p\left(B^*, e_H\right)\right)}{B^* + \left(\frac{c_H}{p(B^*,e_H)-p(B^*,e_L)}\right)\left(1 - p\left(B^*, e_H\right)\right)}$$

and

$$\frac{\partial A_2}{\partial \theta} = \underbrace{\frac{B^* - \left(\frac{c_H}{p(B^*,e_H)-p(B^*,e_L)}\right) p\left(B^*, e_H\right)}{B^* + \left(\frac{c_H}{p(B^*,e_H)-p(B^*,e_L)}\right)\left(1 - p\left(B^*, e_H\right)\right)}}_{+} + \frac{\partial A_2}{\partial B^*} \underbrace{\frac{\partial B^*}{\partial \theta}}_{+}$$

Since

$$\frac{\partial A_2}{\partial B^*} = (B - M)\left(\frac{\partial M}{\partial B} p_H + (B - M) + p_h M p_H' + p_H' M\right) > 0$$

where

$$M = \frac{c_H}{p\left(B^*, e_H\right) - p\left(B^*, e_L\right)}$$

and we denote $p_H = p\left(B^*, e_H\right)$

Therefore, $\frac{\partial A_2}{\partial \theta} > 0$ ■

**Proof. Lemma** 14

We have

$$A_1 = B^* - p\left(B^*, e_H\right)\left(\theta - \frac{c_H}{p\left(B^*, e_H\right) - p\left(B^*, e_L\right)}\right)$$

And

$$\widetilde{A_1} = \widetilde{B}^* - \widetilde{p}\left(\widetilde{B}^*, \widetilde{e_H}\right)\left(\theta - \frac{c_H}{p\left(\widetilde{B}^*, \widetilde{e_H}\right) - p\left(\widetilde{B}^*, \widetilde{e_L}\right)}\right)$$

Since $F(p) = \widetilde{p}$ is a homothetic function, $B^* = \widetilde{B}^*$ and $e_j = \widetilde{e_j}$. Besides, $F' > 0$, we have

$$\widetilde{p}\left(\widetilde{B}^*, \widetilde{e_H}\right) = \lambda^n p\left(B^*, e_H\right) > p\left(B^*, e_H\right)$$

where $\lambda > 1$ Thus,

$$\begin{aligned}
& A_1 - \widetilde{A_1} \\
= & \underbrace{\left(B^* - \widetilde{B}^*\right)}_{=0} - \theta \underbrace{\left(p\left(B^*, e_H\right) - \widetilde{p}\left(\widetilde{B}^*, \widetilde{e_H}\right)\right)}_{-} \\
& + \underbrace{\left[\left(p\left(B^*, e_H\right) \frac{c_H}{p\left(B^*, e_H\right) - p\left(B^*, e_L\right)}\right) - \left(\widetilde{p}\left(\widetilde{B}^*, \widetilde{e_H}\right) \frac{c_H}{p\left(\widetilde{B}^*, \widetilde{e_H}\right) - p\left(\widetilde{B}^*, \widetilde{e_L}\right)}\right)\right]}_{=0} \\
> & \ 0
\end{aligned}$$

Therefore, $A_1 > \widetilde{A_1}$ ■

**Proof. Lemma** 15

We have

$$A_2 = \frac{\theta\left(B^* - \left(\frac{c_H}{p(B^*, e_H) - p(B^*, e_L)}\right) p\left(B^*, e_H\right)\right)}{B^* + \left(\frac{c_H}{p(B^*, e_H) - p(B^*, e_L)}\right)\left(1 - p\left(B^*, e_H\right)\right)}$$

$$\begin{aligned}
\widetilde{A_2} & = \frac{\theta\left(\widetilde{B}^* - \left(\frac{c_H}{p\left(\widetilde{B}^*, \widetilde{e_H}\right) - p\left(\widetilde{B}^*, \widetilde{e_L}\right)}\right) p\left(\widetilde{B}^*, \widetilde{e_H}\right)\right)}{\widetilde{B}^* + \left(\frac{c_H}{p\left(\widetilde{B}^*, \widetilde{e_H}\right) - p\left(\widetilde{B}^*, e_L\right)}\right)\left(1 - p\left(\widetilde{B^*}, \widetilde{e_H}\right)\right)} \\
& = \frac{\lambda^n\left[\theta\left(B^* - \left(\frac{c_H}{p(B^*, e_H) - p(B^*, e_L)}\right) p\left(B^*, e_H\right)\right)\right]}{\lambda^n\left[B^* + \left(\frac{c_H}{p(B^*, e_H) - p(B^*, e_L)}\right)\left(\frac{1}{\lambda^n} - p\left(B^*, e_H\right)\right)\right]} \\
& > A_2
\end{aligned}$$

since

$$\left(\frac{1}{\lambda^n} - p\left(B^*, e_H\right)\right) < \left(1 - p\left(B^*, e_H\right)\right)$$

■

**Proof. Proposition** 16

The proof follows from Lemma 12, Lemma 13, Lemma 14 and Lemma 15. Since $\theta$ and probability function $p\left(B, e_j\right)$ are the sources that affect the risk-neutral firms' profit-to-asset ratios, If $\theta$ increases, combine Lemma 12 and Lemma 13, we can conclude that the probability of a firm using convertible debt contract increases if the firms' profit-to-asset ratio increases since $A_1$ decrease and $A_2$ increases.

If $p\left(B, e_j\right)$ becomes $\widetilde{p}\left(B, e_j\right)$, from Lemma 14 and Lemma 15, $A_1, A_2$ become $\widetilde{A}_1$, $\widetilde{A}_2$ respectively, and

$$prob\left(A \in \left[A_1, A_2\right]\right) < prob\left(A \in \left[\widetilde{A_1}, \widetilde{A_2}\right]\right)$$

Hence, in sum, an increase in firms' profit-to-asset ratio leads to an increase in the probability of a firm using convertible debt. ■

**Proof. Lemma**17

If firms are risk averse, optimal contracts must solve

$$\max_{B,e,s_1,s_2} p(B,e)\,U(\theta - s_1 + A) + (1 - p(B,e))\,U(-s_2 + A) - c(e)$$

s.t.

$$p(B,e)\,s_1 + (1 - p(B,e))\,s_2 - B = 0$$

$$\theta - s_1 + A \geq 0$$

$$-s_2 + A \geq 0$$

This problem is equivalent to

$$\max_{B,e,s_2} p(B,e)\,U\left(\theta - \frac{B - (1 - p(B,e))\,s_2}{p(B,e)} + A\right) + (1 - p(B,e))\,U(-s_2 + A) - c(e)$$

s.t.

$$\theta - \frac{B - (1 - p(B,e))\,s_2}{p(B,e)} + A \geq 0$$

$$-s_2 + A \geq 0$$

We derive the first order conditions for both $B$ and $s_2$:

$$[B]: \quad p'_B\left(U\left(\theta - \frac{B - (1 - p(B,e))\,s_2}{p(B,e)} + A\right) - U(-s_2 + A)\right) \tag{1}$$

$$+pU'\left(\theta - \frac{B - (1 - p(B,e))\,s_2}{p(B,e)} + A\right)\left(-\frac{\partial\left(\frac{B-(1-p(B,e))s_2}{p(B,e)}\right)}{\partial B}\right) \tag{2}$$

$$= \quad 0 \tag{3}$$

$$[s_2] \quad : \quad pU'\left(\theta - \frac{B - (1 - p(B,e))\,s_2}{p(B,e)} + A\right)\frac{\partial\left(\frac{B-(1-p(B,e))s_2}{p(B,e)}\right)}{\partial s_2} \tag{4}$$

$$-(1-p)\,U'(-s_2 + A) \tag{5}$$

$$< \quad 0 \tag{6}$$

From equation 8, we can show that

$$-\frac{\frac{\partial^2 EU}{\partial B^2}}{\frac{\partial^2 EU}{\partial B \partial s_2}} > 0$$

consequently, $\frac{\partial B}{\partial s_2} > 0$.

Since $s_2$ under convertible debt is higher than $s_2$ under equity contracts, hence, $B^{CD} > B^E$ ■

**Proof. Proposition**18

We can show that given $B^E$, the firms' expected utility is higher under convertible debt contracts than under equity contracts, iff

$$\begin{aligned} & p\left(B^E, e_H\right) U\left(\theta - \frac{B^E - \left(1 - p\left(B^E, e_H\right)\right) A}{p\left(B^E, e_H\right)} + A\right) \\ \geq \; & p\left(B^E, e_H\right) U\left(\theta - \frac{B^E \theta}{p\left(B^E, e_H\right)\theta + \left(1 - p\left(B^E, e_H\right)\right) A} + A\right) \\ & + \left(1 - p\left(B^E, e_H\right)\right) U\left(-\frac{B^E A}{p\left(B^E, e_H\right)\theta + \left(1 - p\left(B^E, e_H\right)\right) A} + A\right) \end{aligned}$$

This is equivalent to

$$\frac{U\left(\theta - \frac{B^E - (1 - p(B^E, e_H))A}{p(B^E, e_H)} + A\right) - U\left(\theta - \frac{B^E \theta}{p(B^E, e_H)\theta + (1 - p(B^E, e_H))A} + A\right)}{U\left(-\frac{B^E A}{p(B^E, e_H)\theta + (1 - p(B^E, e_H))A} + A\right)} \tag{7}$$

$$< \frac{\left(\theta - \frac{B^E - (1 - p(B^E, e_H))A}{p(B^E, e_H)} + A\right) - \left(\theta - \frac{B^E \theta}{p(B^E, e_H)\theta + (1 - p(B^E, e_H))A} + A\right)}{-\frac{B^E A}{p(B^E, e_H)\theta + (1 - p(B^E, e_H))A} + A} \tag{8}$$

$$< \frac{1 - p\left(B^{E, e_H}\right)}{p\left(B^E, e_H\right)} \tag{9}$$

The sufficient condition for the inequality above held is $p\left(B^{E, e_H}\right) > \frac{1}{2}$. Since if $p\left(B^{E, e_H}\right) > \frac{1}{2}$,

$$\begin{aligned} & p\left(B^E, e_H\right) \underbrace{\left\{ \begin{array}{c} \left[\frac{B^E \theta}{p(B^E, e_H)\theta + (1 - p(B^E, e_H))A} - \frac{B^E - (1 - p(B^E, e_H))A}{p(B^E, e_H)}\right] \\ -\left(-\frac{B^E A}{p(B^E, e_H)\theta + (1 - p(B^E, e_H))A} + A\right) \end{array} \right\}}_{(i)} \\ < \; & \left(-\frac{B^E A}{p\left(B^E, e_H\right)\theta + \left(1 - p\left(B^E, e_H\right)\right) A}\right) + A \end{aligned}$$

since term (i) is always negative if

$$p\left(B^E, e_H\right) > \frac{1}{2}$$

Therefore, inequality 8 holds. Besides, from Lemma 17, we have shown that in equilibrium, $B^E < B^{CD}$.

As a result, we have shown that given $B^E$, firms' expected utility is higher under convertible debt than under equity

$$EU^{CD}\left(B^E\right) > EU^E\left(B^E\right)$$

Together with the result $B^E < B^{CD}$ form Lemma 17, $B^{CD}$ maximizes the firms' expected utility under convertible debt, we derive

$$EU^{CD}\left(B^{CD}\right) > EU^{CD}\left(B^E\right) > EU^E\left(B^E\right)$$

and the result is shown. ■